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Interpolating coherent states for Heisenberg–Weyl and single-photon $SU(1,1)$ algebras

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Abstract

New quantal states which interpolate between the coherent states of the Heisenberg–Weyl (W_3) and $SU(1,1)$ algebras are introduced. The interpolating states are coherent states of a closed and symmetric algebra containing a parameter k which takes values from zero to one. The elements of the algebra are realized in terms of operator-valued functions of creation and annihilation operators. The realization is such that the operators become those of the W_3 algebra when $k = 0$ and of the $SU(1,1)$ algebra when k is unity. The overcompleteness of the interpolating coherent states is established by constructing suitable integration measures. Differential operator representations for the elements of the algebra are given along with the relevant spaces of entire functions. A nonsymmetric set of operators to realize the W_3 algebra is presented and the relevant coherent states are studied. Physical properties such as squeezing and photon statistics of the interpolating coherent states are investigated.

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1. Introduction

An extremely useful mathematical framework for dealing with continuous symmetries is the theory of Lie groups and Lie algebras. In the context of quantum optics, the use of group theory has been very prominent ever since the discovery of the coherent states of an electromagnetic field [1–4]. The usual coherent states are the unitarily displaced vacuum state of the harmonic oscillator. The unitary displacement is effected by the displacement operator $D(\alpha)$ given by $\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, $\hat{a}^\dagger$ and $\hat{a}$ being the creation and annihilation operators, respectively. Each coherent state is characterized by a complex number α and the state is expressed in terms of Fock (number) states as

$$|\alpha\rangle = D(\alpha)|0\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha}{\sqrt{n!}}|n\rangle. \tag{1}$$

The algebra relevant to these states is the Heisenberg–Weyl algebra W_3, generated by the operators $\hat{a}$ and $\hat{a}^\dagger$, and the identity operator I. These three operators form a closed algebra since $[\hat{a}, \hat{a}^\dagger] = I$ and I commutes with both $\hat{a}$ and $\hat{a}^\dagger$. The Heisenberg–Weyl algebra can be extended to W_4 by including the number operator $\hat{a}^\dagger\hat{a}$ and the algebra is still closed. It turns out that the states $|\alpha\rangle$ are the eigenstates of $\hat{a}$, one of the elements of the algebra. Such eigenstates are algebraic coherent states. States obtained, as in equation (1), by a unitary transformation are said to be group-theoretic coherent states or coherent states in the sense of Perelomov. The most notable feature of coherent states is their overcompleteness and it is mathematically expressed as

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha|\alpha\rangle\langle\alpha| = I. \tag{2}$$

The integration is over the entire complex plane and the integration measure $\mathrm{d}^2\alpha$ is $\mathrm{d}(\mathrm{Re}(\alpha))\,\mathrm{d}(\mathrm{Im}(\alpha))$.

The two-photon operators $\hat{a}^2$ and $\hat{a}^{\dagger 2}$ and the number operator $\hat{a}^\dagger\hat{a}$ form a closed algebra which is identical to the well-known $SU(1,1)$ algebra of three operators K_0, K_+ and K_- which satisfy

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2K_0. \tag{3}$$

In the case of two-photon operators, K_0, K_+ and K_- are identified with $(2\hat{a}^\dagger\hat{a} + 1)/4$, $\hat{a}^{\dagger 2}/2$ and $\hat{a}^2/2$, respectively. The algebraic coherent states for this realization are the even and odd coherent states [5]. The group-theoretic coherent states are the squeezed vacuum and first excited states [6–8]. It is, however, possible to realize $SU(1,1)$ algebra with deformed single-photon operators [9–14]. Here, deformation implies that the generators are multiplied by an operator-valued function of the number operator. Consider the Holstein–Primakoff realization:

$$K_0 = \hat{a}^\dagger\hat{a} + j \qquad K_- = \sqrt{\hat{a}^\dagger\hat{a} + 2j}\,\hat{a} \qquad K_+ = \hat{a}^\dagger\sqrt{\hat{a}^\dagger\hat{a} + 2j}. \tag{4}$$

The Casimir invariant for this realization is $j(j-1)$. The operator realizations indeed satisfy the $SU(1,1)$ algebra for all values of j, but they are not two-photon operators. Here, j is a constant and we set it equal to $\frac{1}{2}$ in the following discussion. The algebraic coherent states, defined as the eigenstates of K_-, for this realization are

$$|\alpha, 1\rangle\rangle = N\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}|n\rangle. \tag{5}$$

The normalization constant N is $1/\sqrt{I_0(2|\alpha|)}$ and I_0 is the Bessel function of second kind of order zero [15]. These coherent states correspond to the 'discrete' representation where the parameter j is $1/2$. In general, j can be either an integer or a half-integer for discrete representation. Eigenstates of deformed annihilation operators are termed 'nonlinear coherent states' as they can be thought of as the coherent states of an oscillator with energy-dependent frequency [16–18].

The group-theoretic coherent states are constructed as

$$\begin{aligned}|\alpha, 1\rangle_p &= \exp(\alpha K_+ - \alpha^* K_-)|0\rangle \\ &= \frac{1}{\sqrt{1-|\zeta|^2}}\sum_{n=0}^{\infty}\zeta^n|n\rangle.\end{aligned} \tag{6}$$

The parameter ζ is a function of $\alpha = |\alpha|\exp(\mathrm{i}\theta)$ and the relation is $\zeta = \exp(\mathrm{i}\theta)\tanh(|\alpha|)$.

The two realizations of $SU(1,1)$ algebra, one in terms of two-photon operators and the other in terms of deformed single-photon operators, are useful in solving the intensity-dependent Jaynes–Cummings model (JCM) [19]. The two-photon realization of $SU(1,1)$ algebra is relevant if the interaction between a single-mode cavity field and a two-level atom is described by

$$\hat{H}_{\mathrm{int}} = g(\sigma_{-}\hat{a}^{\dagger 2} + \sigma_{+}\hat{a}^{2}). \tag{7}$$

The operators $\sigma_{\pm}$ are the raising and lowering operators for the two levels of the atom. For the Holstein–Primikoff realization (with $j = 0$), the relevant interaction is

$$\hat{H}_{\mathrm{int}} = g(\sigma_{-}\hat{a}^{\dagger}\sqrt{\hat{a}^{\dagger}\hat{a}} + \sigma_{+}\sqrt{\hat{a}^{\dagger}\hat{a}}\hat{a}). \tag{8}$$

The interaction describes single-photon processes with nonlinear coupling between the atom and the field. The latter case has been extensively studied in the context of JCM which exhibits complete periodicity in population inversion of atomic levels and fields with infinite statistics [20].

A more general case of interaction is

$$\hat{H}_{\mathrm{int}} = g(\sigma_{-}\hat{a}^{\dagger}\sqrt{k\hat{a}^{\dagger}\hat{a}+1} + \sigma_{+}\sqrt{k\hat{a}^{\dagger}\hat{a}+1}\hat{a}). \tag{9}$$

In the absence of nonlinear coupling, obtained by setting $k = 0$, this Hamiltonian describes the usual JCM. In the following section, we consider operators which are relevant for the Hamiltonian given in equation (9). It is shown that the operators satisfy the W_3 or $SU(1,1)$ algebra depending on whether k is zero or unity. In section 3, we construct algebraic coherent states for the interpolating algebra and study the properties of the states as a function of k. The overcompleteness of the states is proven. Section 4 deals with group-theoretic coherent states for the interpolating algebra and their properties. Properties such as energy fluctuations and quadrature squeezing are studied. In section 5, we introduce a nonsymmetric set of operators to realize the W_3 algebra and study the relevant coherent states.

2. Generalization of single-photon $SU(1,1)$ coherent states

We introduce an additional parameter k (non-negative and less than or equal to unity) in the symmetric set of operators defined in equation (4) such that the $SU(1,1)$ realization is obtained when $k = 1$. The symmetric set of operators is

$$A_0 = k\hat{a}^{\dagger}\hat{a} + \tfrac{1}{2} \qquad A_{-} = \sqrt{k\hat{a}^{\dagger}\hat{a}+1}\,\hat{a} \qquad A_{+} = \hat{a}^{\dagger}\sqrt{k\hat{a}^{\dagger}\hat{a}+1}. \tag{10}$$

These operators are closed under commutation, and we have

$$[A_0, A_{\pm}] = \pm k A_{\pm} \qquad [A_{+}, A_{-}] = -2A_0. \tag{11}$$

The Casimir invariant of this closed algebra is $A_0^2 - (k/2)\{A_{-}, A_{+}\} = \frac{1}{2}\left(\frac{1}{2} - k\right)$, where $\{A_{-}, A_{+}\}$ stands for anticommutation of the two operators. Successive eigenvalues of the Casimir operator differ by k. This has to be compared with that of the canonical $SU(1,1)$ algebra where successive eigenvalues differ by unity.

Two important limiting cases of the above commutation relations are obtained when k takes the values zero and unity, respectively. In the former case, the algebra reduces to W_3 and in the latter case it becomes the $SU(1,1)$ algebra. Thus, the algebra interpolates between the $SU(1,1)$ and W_3 algebras. The fact that one algebra can be obtained from another algebra is known as 'contraction' and the procedure to go from $SU(1,1)$ to W_3 is known [21–25]. What we have presented here is a *realization* of an algebra which has W_3 and $SU(1,1)$ as the limiting cases, i.e. the aforementioned realization interpolates the W_3 and $SU(1,1)$ algebras. We call the algebra defined in equation (11) the interpolating algebra. It is worth mentioning that the binomial states of the harmonic oscillator interpolate between the number states and coherent states [26]. Recently, the eigenstates of the operator $\sqrt{\eta}\hat{a}^\dagger\hat{a} + \sqrt{(1-\eta)}\hat{a}$, the sum of two operators with number states and coherent states as respective eigenstates, were studied [27]. In the case of $\eta = 0$, the operator becomes $\hat{a}$ and the eigenstates are the coherent states. In the limit of $\eta \to 1$, the eigenstates are those of the number operator, i.e. number states. The eigenstates corresponding to the in-between values of η are said to interpolate between the coherent states and number states. However, note that the operators considered in the present work are NOT in the form of an addition of operators of W_3 and $SU(1,1)$ algebras.

3. Algebraic coherent states

The algebraic coherent states for the interpolating algebra are defined by

$$A_-|\alpha, k\rangle = \alpha|\alpha, k\rangle. \tag{12}$$

Using the realization of A_- given in equation (10), the number state expansion is

$$|\alpha, k\rangle = N_k \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!k^n\left(\frac{1}{k}\right)_n}}|n\rangle. \tag{13}$$

The states are normalizable for all values of α and the normalization constant N_k is given by

$$N_k^2 = \left(\frac{|\alpha|}{\sqrt{k}}\right)^{1-\frac{1}{k}} \Gamma\left(\frac{1}{k}\right) I_{1-\frac{1}{k}}\left(\frac{2|\alpha|}{\sqrt{k}}\right). \tag{14}$$

The symbol $\left(\frac{1}{k}\right)_n$ is the Pochammer notation for the product $\Pi_{j=1}^{n}\left(\frac{1}{k}+j-1\right)$ and $\left(\frac{1}{k}\right)_0 = 1$ [15]. As expected, in the limit of k becoming zero the states $|\alpha, k\rangle$ become the usual coherent states $|\alpha\rangle$.

3.1. Completeness of algebraic coherent states

In this section, we prove the completeness relation for the states $|\alpha, k\rangle$. We need to show that

$$\frac{1}{\pi}\int \mathrm{d}\mu|\alpha, k\rangle\langle\alpha, k| = I \tag{15}$$

for some suitable integration measure μ. This problem naturally leads to the *problem of moments* wherein it is required to construct a probability distribution from the knowledge of its moments [28, 29]. Substituting the number state expansion in equation (13) into equation (15), the lhs of the latter equation becomes

$$\mathrm{lhs} = \frac{1}{\pi}\int \mathrm{d}\mu N_k^2 \sum_{n,m=0}^{\infty} \frac{\alpha^n\alpha^{*m}}{\sqrt{n!m!k^{n+m}\left(\frac{1}{k}\right)_n\left(\frac{1}{k}\right)_m}}|n\rangle\langle m|. \tag{16}$$

On substituting $\alpha = r\exp(\mathrm{i}\theta)$ and setting $\mathrm{d}\mu = \frac{\rho(r)r}{N_k^2}\,\mathrm{d}r\,\mathrm{d}\theta$, the completeness relation becomes

$$I = \frac{1}{\pi}\int \rho(r)r\,\mathrm{d}r \sum_{n=0}^{\infty} \frac{r^{2n}}{n!k^n\left(\frac{1}{k}\right)_n}|n\rangle\langle n|. \tag{17}$$

For the equation to be valid, the condition is

$$\int \rho(x)x^n\,\mathrm{d}x = \frac{2\Gamma(n+1)\Gamma\left(\frac{1}{k}+n\right)}{\Gamma\left(\frac{1}{k}\right)} \tag{18}$$

with $x = r^2$. The equation gives the moments of the unknown function $\rho(x)$ and from the known moments the form of the function has to be inferred. By comparing with the standard formula [30]

$$\int_0^\infty x^{s-1} x^{(1/k-1)/2} K_{\frac{1}{k}-1}(2\sqrt{x}/k)\,\mathrm{d}x = \frac{1}{2}k^s \Gamma\left(s + \frac{1}{k} - 1\right)\Gamma(s) \tag{19}$$

we infer

$$\rho(r) = \frac{2}{k\Gamma\left(\frac{1}{k}\right)} r^{\left(\frac{1}{k}-1\right)} K_{\frac{1}{2}\left(\frac{1}{k}-1\right)}\left(\frac{2}{k}r\right). \tag{20}$$

Here, K_ν is the modified Bessel function of order ν [15]. Thus, the states $|\alpha, k\rangle$ provide a resolution of identity. The inner product between two states, say $|\alpha, k\rangle$ and $|\beta, k\rangle$, is

$$\langle \alpha, k|\beta, k\rangle = (\alpha^*\beta)^{\frac{k-1}{2k}} \frac{I_{\frac{1-k}{k}}(2\sqrt{\alpha^*\beta/k})}{\sqrt{|\alpha\beta|^{\frac{k-1}{k}} I_{\frac{1-k}{k}}(2|\beta|/\sqrt{k}) I_{\frac{1-k}{k}}(2|\alpha|/\sqrt{k})}}. \tag{21}$$

As two states corresponding to two different values of α are not orthogonal, the states $|\alpha, k\rangle$ are overcomplete.

The uniqueness of the weight function $\rho(r)$ is guaranteed if the moments μ_n ($n = 0, 1, 2, \ldots$) satisfy the sufficient condition

$$\sum_{n=0}^{\infty} \mu_n{}^{-\frac{1}{2n}} = \infty. \tag{22}$$

For the present case, the moments are given by equation (18) and they satisfy the sufficient condition. Therefore, the weight function $\rho(r)$ is unique. Recently, a variety of coherent states have been constructed based on the solution of the moments problem using the tabulated inverse Mellin transforms [31].

Any harmonic oscillator state $|\psi\rangle$ can be expanded in terms of the overcomplete set of states $|\alpha, k\rangle$ as

$$|\psi\rangle = \frac{1}{\pi}\int \mathrm{d}\mu\, f(\alpha^*)|\alpha, k\rangle. \tag{23}$$

The function $f(\alpha^*)$ is $\langle \alpha, k|\psi\rangle$, the projection of the state $|\psi\rangle$ on the eigenstates of A_-. In the space of functions $N_k^{-1}(|\alpha|) f(\alpha^*)$, the generators A_-, A_+ and A_0 are represented as

$$A_+ = \alpha^* \qquad A_- = \frac{\mathrm{d}}{\mathrm{d}\alpha^*} + k\alpha^* \frac{\mathrm{d}^2}{\mathrm{d}\alpha^{*2}} \qquad A_0 = k\alpha^* \frac{\mathrm{d}}{\mathrm{d}\alpha^*} + \frac{1}{2}. \tag{24}$$

3.2. Squeezing and occupation-number statistics

The two limiting cases of $|\alpha, k\rangle$ are the coherent states $|\alpha\rangle$ and the states $|\alpha, 1\rangle$ corresponding to k becoming zero and unity, respectively. The states corresponding to other values of k interpolate between the two limiting cases. The term 'interpolating states' seems appropriate as the properties of the states are intermediate between those of the limiting cases. For instance, the coherent states are Poissonian, meaning that the variance and mean of the number distribution are equal. The single-photon $SU(1, 1)$ coherent states are sub-Poissonian, i.e. the variance is less than the mean for their number distribution. A quantitative measure for the deviation from Poissonian behaviour is the Q-parameter [32], defined as

$$Q = \frac{\langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a}\rangle - \langle \hat{a}^\dagger \hat{a}\rangle^2}{\langle \hat{a}^\dagger \hat{a}\rangle}. \tag{25}$$

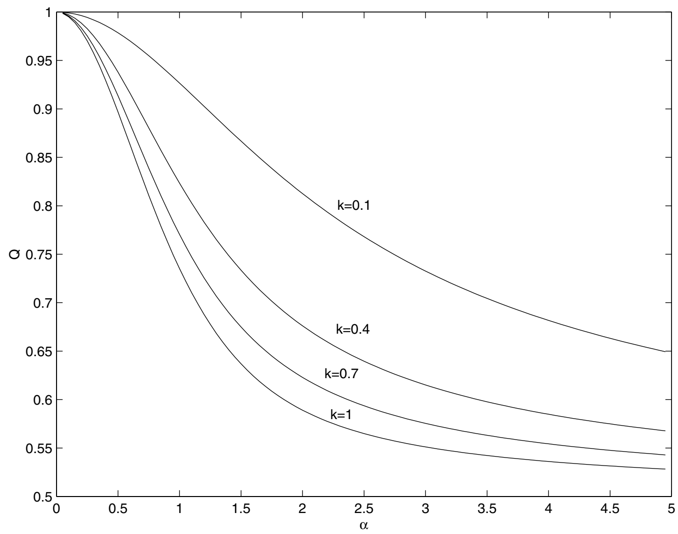

Figure 1. Variation of the Q-parameter as a function of α for the states $|\alpha, k\rangle$.

For all states exhibiting sub-Poissonian statistics, the Q-parameter is less than unity. In figure 1, the Q-parameter is shown as a function of $|\alpha|$ for different values of k. The states $|\alpha, k\rangle$ are sub-Poissonian for $0 < k \leqslant 1$. The states corresponding to arbitrary k ($\neq 0$) are also sub-Poissonian, however, the value of Q-parameter lies between those of the coherent states of the $SU(1,1)$ and W_3 algebras.

The algebraic coherent states exhibit squeezing in both the field quadratures, namely,

$$\hat{x} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} \qquad \text{and} \qquad \hat{p} = \frac{\hat{a} - \hat{a}^\dagger}{\mathrm{i}\sqrt{2}}. \tag{26}$$

For the coherent states $|\alpha\rangle$, the uncertainties in x and p are the same as those of the vacuum state. In the case of $|\alpha, k\rangle$, the squeezing in the x quadrature increases with both α and k. The dependence is depicted in figures 2(a)–(d), where the variation of Δx with α is shown for various values of k. It is interesting to note that the uncertainty profiles are symmetric under $\alpha \to -\alpha$. This can be understood as follows. The symbol $\langle \cdots \rangle$ stands for the expectation value in the states $|\alpha, k\rangle$. In terms of the operators $\hat{a}$ and $\hat{a}^\dagger$, the uncertainty in x is

$$(\Delta x)^2 = \tfrac{1}{2}[1 + 2\langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}^{\dagger 2}\rangle + \langle \hat{a}^2\rangle - \langle \hat{a}^\dagger\rangle^2 - \langle \hat{a}\rangle^2 - 2\langle \hat{a}^\dagger\rangle\langle \hat{a}\rangle \tag{27}$$

and that in p is

$$(\Delta p)^2 = \tfrac{1}{2}[1 + 2\langle \hat{a}^\dagger \hat{a}\rangle - \langle \hat{a}^{\dagger 2}\rangle - \langle \hat{a}^2\rangle + \langle \hat{a}^\dagger\rangle^2 + \langle \hat{a}\rangle^2 - 2\langle \hat{a}^\dagger\rangle\langle \hat{a}\rangle. \tag{28}$$

When $\alpha \to \exp(\mathrm{i}\theta)\alpha$, where $0 \leqslant \theta \leqslant 2\pi$, we have

$$\langle \hat{a}^\dagger\rangle \to \exp(-\mathrm{i}\theta)\langle \hat{a}^\dagger\rangle \qquad \langle \hat{a}^{\dagger 2}\rangle \to \exp(-\mathrm{i}2\theta)\langle \hat{a}^{\dagger 2}\rangle$$

and

$$\langle \hat{a}^\dagger \hat{a}\rangle \to \langle \hat{a}^\dagger \hat{a}\rangle.$$

The transformation $\alpha \to -\alpha$ corresponds to $\theta = \pi$. On using the transformed expressions in equations (27) and (28), we get the uncertainties in x and p for the state $|\alpha\rangle$ to be the same as the respective values for the state $|-\alpha, k\rangle$. Another interesting transformation is $\alpha \to \mathrm{i}\alpha$, which corresponds to rotation by $\pi/2$. Under this transformation, the uncertainty in x for the state $|\alpha, k\rangle$ goes over to uncertainty in p for the state $|\mathrm{i}\alpha\rangle$. The above discussion implies that the knowledge of variance in one of the quadratures for all values of α also gives the magnitude

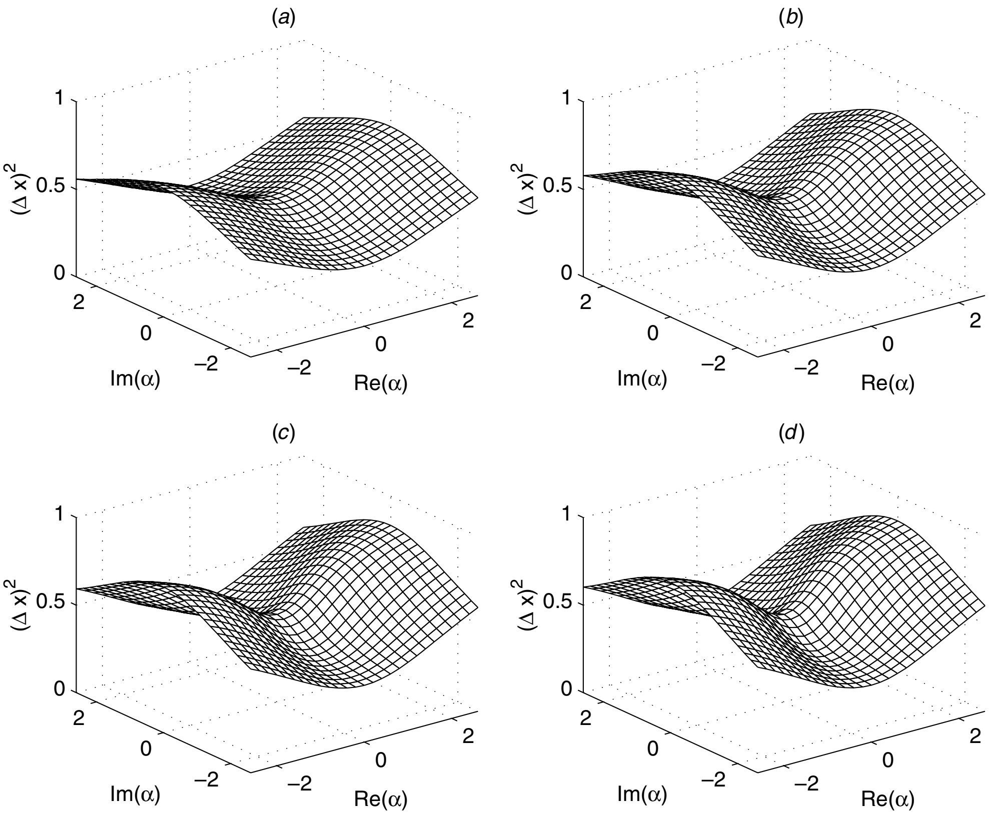

Figure 2. Uncertainty in x for the algebraic coherent states for all values of $|\alpha| \leqslant 2.5$. (a) $k = 0.25$, (b) $k = 0.5$, (c) $k = 0.75$ and (d) $k = 1.0$. Regions of squeezing correspond to those points where the uncertainty falls below 0.5, the coherent state value.

of the fluctuation in the other quadrature. The uncertainty profile for the p-quadrature is the same as that of the x, except for a rotation of $\pi/2$ about the axis labelled Δx. Another observation is that the x-quadrature exhibits reduced fluctuations for real α and increased fluctuations for imaginary α.

The symmetries exhibited in the uncertainty profiles are not restricted to the states $|\alpha, k\rangle$. They are generic to any state $|S, \alpha\rangle$, characterized by a complex number α, which has the number state expansion

$$|S, \alpha\rangle = \sum_{n=0}^{\infty} \alpha^n S_n |n\rangle \tag{29}$$

where the coefficients S_n are real.

The states $|\alpha, k\rangle$ are nonclassical as they exhibit squeezing in the quadratures and exhibit sub-Poissonian photon statistics. Hence the Wigner function should become negative somewhere on the complex plane. The Wigner function, obtained using the method given in [33], is given by

$$W(z) = \frac{2\exp(-zz^*)}{\pi} N_k^2 \sum_{n,m=0}^{\infty} \frac{\alpha^n \alpha^{*m}}{\sqrt{k^{n+m}\left(\frac{1}{k}\right)_n \left(\frac{1}{k}\right)_m}} \sum_{l=0}^{\min(m,n)} \frac{2^{n+m-2l} z^{n-l} z^{*m-l} (-)^l}{l!(n-l)!(m-l)!} \tag{30}$$

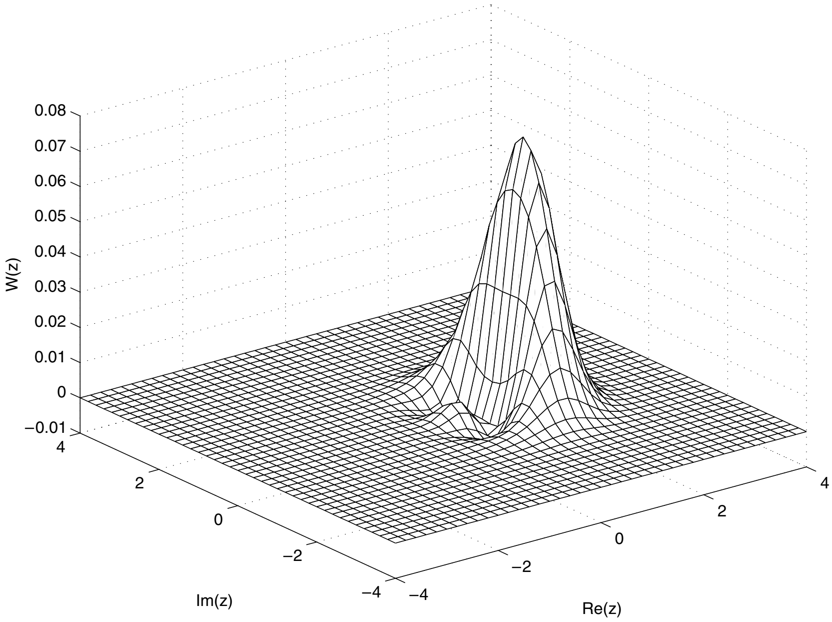

Figure 3. Plot of the Wigner function $W(z)$ for the state $|2.5, 0.5\rangle$.

We have plotted in figure 3 the Wigner function for the state $|2.5, 0.5\rangle$ which exhibits squeezing in x-quadrature (see figure 2(b)). As expected, there are valleys of negative values of the Wigner function.

4. Group-theoretic coherent states

The group-theoretic coherent states for the algebra of operators defined in equation (10) are constructed by the action of the unitary operator $\exp(\alpha A_+ - \alpha^* A_-)$ on the vacuum state $|0\rangle$. Denoting these states as $|\alpha, k\rangle_p$, where the suffix p stands for the 'Perelomov state', we have

$$|\alpha, k\rangle_p = \exp(\alpha A_+ - \alpha^* A_-)|0\rangle. \tag{31}$$

To get the number state expansion for the rhs of the above equation, we use the following disentangled form, derived using the method described in [34, 35], for the unitary operator,

$$\begin{aligned} \exp(\alpha A_+ - \alpha^* A_-) &= \exp(\beta A_+)\exp(\gamma A_0)\exp(\delta A_-) \\ \beta &= \frac{\exp(\mathrm{i}\theta)}{\sqrt{k}}\tanh(\lambda\sqrt{k}) \\ \gamma &= -\frac{2}{k}\log[\cosh(\lambda\sqrt{k})] \\ \delta &= -\beta^*. \end{aligned} \tag{32}$$

In the above expressions, λ and θ are respectively the modulus and argument of α. In the limit of $k \to 0$, the above expression reduces to the familar form $\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) = \exp(-\lambda^2/2)\exp(\alpha\hat{a}^\dagger)\exp(-\alpha^*\hat{a})$.

The disentangled form of $\exp(\alpha A_+ - \alpha^* A_-)$, as given by equation (32), when substituted in the rhs of equation (31) gives the number state expansion for the state $|\alpha, k\rangle_p$ as

$$|\alpha, k\rangle_p = (1 - k|\beta|^2)^{\frac{1}{2k}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} \sqrt{k^n \left(\frac{1}{k}\right)_n}\,|n\rangle. \tag{33}$$

The states are normalizable for all values of α as $k|\beta|^2 = \tanh^2(\sqrt{k}\lambda) \leqslant 1$ for any α. The disentangled form implies that the states can also be obtained by the action of $\exp(\beta A_+)$ on the vacuum state $|0\rangle$ and normalizing the resultant state. Of course, this is possible only for the vacuum state as A_- annihilates the vacuum and $\exp(\gamma A_0)$ introduces an overall phase. In the limit of $k \to 1$, the states become the well-known phase states [36].

The inner product of $|\beta', k\rangle_p$ with $|\beta, k\rangle_p$ is

$$_p\langle\beta, k|\beta'\rangle_p = [(1-k|\beta|^2)(1-k|\beta'|^2)]^{\frac{1}{2k}}(1-k\beta^*\beta')^{-\frac{1}{k}}. \tag{34}$$

4.1. Completeness of group-theoretic coherent states

The completeness relation for the states $|\alpha, k\rangle_p$ is derived in the same way as for the algebraic coherent states. The resolution of identity by the states $|\alpha, k\rangle_p$ is written as

$$\frac{1-k}{\pi}\int_{|\beta|^2\leqslant\frac{1}{k}} |\alpha, k\rangle_{p\,p}\langle\alpha, k|\frac{\mathrm{d}^2\beta}{1-k|\beta|^2} = I. \tag{35}$$

The range of integration is restricted to a disc of radius $\frac{1}{\sqrt{k}}$ in the complex β-plane. If we use the relation $\sqrt{k}|\beta| = \tanh(\sqrt{k}|\alpha|)$, the finite range of integration in the β-plane goes over to integration over the entire α-plane. The resolution of identity enables us to write an arbitrary state $|\psi\rangle$ in terms of $|\alpha, k\rangle_p$ as

$$|\psi\rangle = \frac{1-k}{\pi}\int \frac{\mathrm{d}\beta}{1-k|\beta|^2} g(\alpha^*)|\alpha, k\rangle_p \tag{36}$$

in which we have used the definition $g^*(\alpha) = \langle\psi|\alpha, k\rangle_p$. In the space of $(1-k|\alpha|^2)^{\frac{1}{2k}} g(\alpha^*)$, the operators of the algebra are

$$A_- = \frac{\mathrm{d}}{\mathrm{d}\alpha^*} \qquad A_+ = k\alpha^{*2}\frac{\mathrm{d}}{\mathrm{d}\alpha^*} + \alpha^* \qquad A_0 = k\alpha^*\frac{\mathrm{d}}{\mathrm{d}\alpha^*} + \frac{1}{2}. \tag{37}$$

In the limit of $k \to 0$, the differential operator realizations of the interpolating algebra in the respective spaces, namely, the Hilbert space which is composed of functions $N_k^{-1} f(\alpha^*)$ and $N_p^{-1} g(\alpha^*)$, yield $\frac{\mathrm{d}}{\mathrm{d}\alpha^*}$, α^* and $\frac{1}{2}$. This is to be expected as the algebraic and group-theoretic coherent states are the same in the limit of vanishing k. The representation space contains the entire functions $\exp(|\alpha|^2/2)\langle\alpha|\psi\rangle$.

The group-theoretic coherent states are always super-Poissonian ($Q > 1$). We have shown, in figure 4, the α-dependence of the Q-parameter. Quadrature squeezing has also been studied for these states. As the states $|\alpha, k\rangle_p$ are normalizable only for $\beta \leqslant 1$, we have studied the squeezing for β lying within the unit circle. Figures 5(a)–(d) give the uncertainty in x as a function of β for $k = 0.25, 0.5, 0.75$ and 1, respectively. As in the case of the algebraic coherent states $|\alpha, k\rangle$, the uncertainty profiles exhibit symmetry when $\alpha \to -\alpha$. Also, the corresponding profiles for p can be obtained by rotating figures 5(a)–(d) by $\pi/2$ about the (Δx)-axis.

5. Coherent states, phase states and W_3 algebra

In this section, we introduce a nonsymmetric set of operators to realize the W_3 algebra and construct the relevant coherent states. Consider the operators A_+, I and $B_- = \frac{1}{\sqrt{1+k\hat{a}^\dagger\hat{a}}}\hat{a}$. These operators satisfy $[B_-, A_+] = I$ for all values of k and provide a realization for the W_3 algebra. The set of operators, however, is not symmetric except when $k = 0$ and in that case we recover the creation and annihilation operators of the harmonic oscillator. The operator

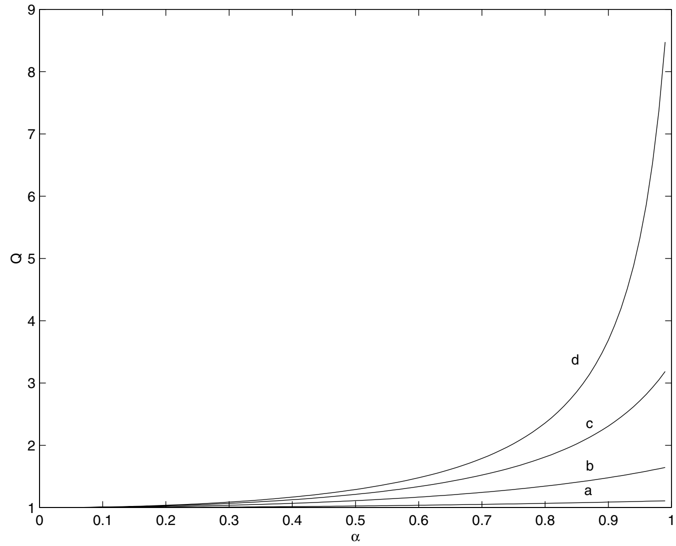

Figure 4. The Q-parameter as a function of α for the states $|\alpha, k\rangle_p$. Different curves correspond to different values of k. (a) $k = 0.25$, (b) $k = 0.5$, (c) $k = 0.75$ and (d) $k = 1.0$.

B_- is constructed using the method of Shanta *et al* [37]. The algebraic coherent states for this algebra are the eigenstates of B_-. Denoting the eigenstates by $|\alpha, k\rangle\rangle$ and using the definition

$$B_-|\alpha, k\rangle\rangle = \alpha|\alpha, k\rangle\rangle \tag{38}$$

the number state expansion is

$$|\alpha, k\rangle\rangle = \frac{1}{(1 - k|\alpha|^2)^{\frac{1}{2k}}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{k^n \left(\frac{1}{k}\right)_n} |n\rangle. \tag{39}$$

The states are normalizable provided $|\alpha|^2 \leqslant 1/k$. As $[B_-, A_+] = I$, the unnormalized eigenstates of B_- can be written as

$$|\alpha, k\rangle\rangle = \exp(\alpha A_+)|0\rangle. \tag{40}$$

These states are identified with $|\alpha, k\rangle_p$ on setting $\alpha = \sqrt{k}\beta$. In the limit of $k \to 0$, the state $|\alpha, k\rangle\rangle$ becomes the coherent state $|\alpha\rangle$ and when $k \to 1$ we get the phase states as eigenstates. For other values of k, the states $|\alpha, k\rangle\rangle$ interpolate between the coherent states and the phase states. Thus, the group-theoretic coherent states for the interpolating algebra have been written as the algebraic coherent states of a different algebra.

Another set of operators which are closed under commutation consists of A_-, I and $B_+ = B_-^\dagger$. These operators are obtained by taking the adjoint of the operators defined in the beginning of this section. The algebraic coherent states for this algebra are the eigenstates of A_- and they have already been discussed in section 3. The relation $[A_-, B_+] = I$ implies that the unnormalized eigenstates of A_- are obtained by a nonunitary deformation of the vacuum state as follows:

$$|\alpha, k\rangle = \exp(\alpha B_+)|0\rangle. \tag{41}$$

The result shows that the algebraic coherent states for the interpolating algebra can be written as a nonunitarily-deformed vacuum state.

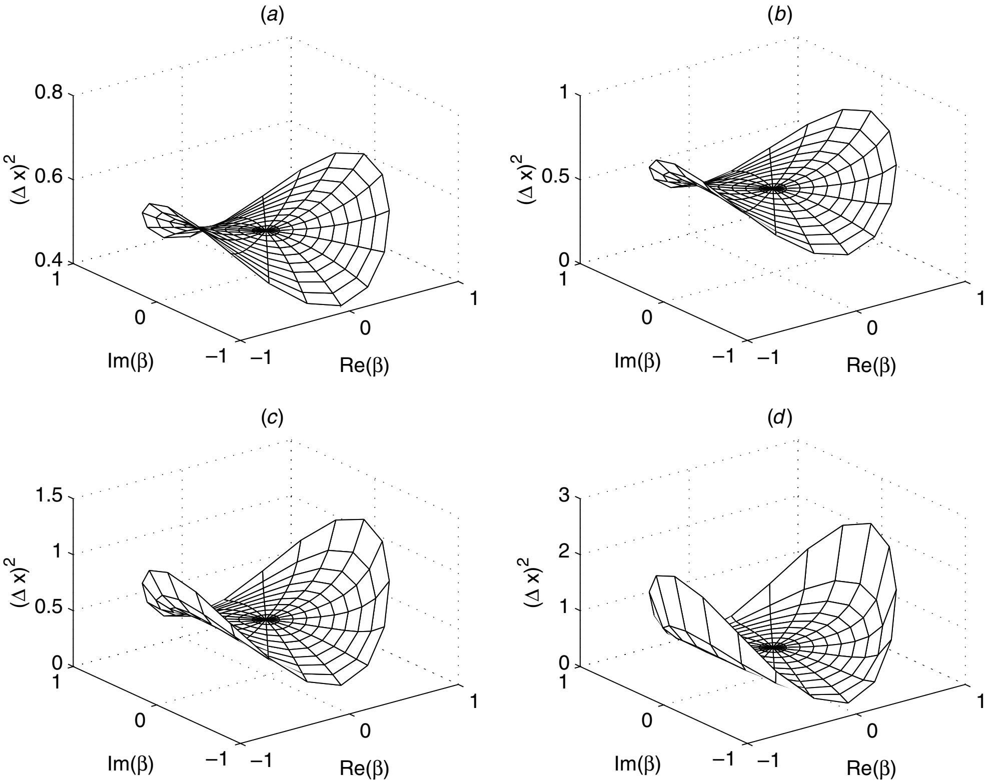

Figure 5. The x-quadrature fluctuations as a function of β. Squeezing occurs for those values of β where fluctuations are less than 0.5. (a) $k = 0.25$, (b) $k = 0.5$, (c) $k = 0.75$ and (d) $k = 1.0$.

6. Summary

The operators $k\hat{a}^\dagger\hat{a} + \frac{1}{2}$, $\sqrt{k\hat{a}^\dagger\hat{a} + 1}\hat{a}$ and $\hat{a}^\dagger\sqrt{k\hat{a}^\dagger\hat{a} + 1}$ form a closed algebra. The parameter k takes all values from zero to unity. When $k = 0$, the operators satisfy W_3 algebra and when $k = 1$ the operators are those of the $SU(1,1)$ algebra. With k being assigned a value between zero and unity, the algebra of the operators interpolates between the W_3 and $SU(1,1)$ algebras. As the interpolating algebra is realized in terms of the harmonic oscillator operators, both algebraic and group-theoretic coherent states for the general algebra have been expanded in the Fock state basis. The states are overcomplete and furnish resolution of identity. Differential operator representation of the elements of the algebra have been constructed and the relevant spaces of entire functions for the representation have been identified. Algebraic as well as group-theoretic coherent states of the interpolating algebra exhibit squeezing in the quadratures. The algebraic coherent states exhibit sub-Poissonian statistics in their occupation-number distribution while the group-theoretic coherent states are super-Poissonian.

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